MEAN VALUES AND DIFFERENTIAL EQUATIONS

BY

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ABSTRACT

We show that the well-known equivalence between the mean-value theorem and harmonicity extends to arbitrary measures of compact support: a continuous function satisfies the generalized mean-value condition (1) with respect to a given measure if and only if it is annihilated by a certain system of homogeneous linear partial differential operators with constant coefficients determined by the measure. Extensions of this result are obtained, primarily in the direction of replacing systems of differential equations by a single equation.

In this paper we are concerned with determining necessary and sufficient conditions for a function to satisfy a generalized mean-value theorem of the form

$$\int u(x + rt)d\mu(t) = 0. \tag{1}$$

Here, μ is a finite complex Borel measure supported in the closed unit ball in $\mathbb{R}^n$, u is a continuous function on some domain $\mathcal{D}$ in $\mathbb{R}^n$, and (1) is required to hold for all $x \in \mathcal{D}$ and $0 < r < \operatorname{dist}(x, \partial\mathcal{D})$.

Special cases of (1) are familiar from potential theory and complex analysis. Thus, if $\mu = \Omega - \delta_0$, where Ω denotes the uniform distribution of total mass 1 on the unit sphere S^{n-1} in $\mathbb{R}^n$ and δ_0 is the (unit) point mass at the origin, (1) becomes the classical mean-value condition characterizing harmonic functions. For $n = 2$ and $d\mu = dz$ (restricted to the unit circle), a necessary and sufficient condition that (1) hold is that u be holomorphic. Here, sufficiency is a trivial consequence of Cauchy's theorem, while necessity follows from a variant of Morera's theorem due to Carleman (see [40]). It was the attempt, on the one hand to explore the close formal connection (already emphasized in [40]) between the Morera-Carleman theorem and the Koebe-Levi-Tonelli converse to the mean-value

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theorem for harmonic functions, and on the other hand to understand the curious if-and-only-if nature of these theorems that led to the present investigation. It will transpire (cf. [23, p. 39]) that this last phenomenon is far from an isolated occurrence: indeed, whole classes of functions defined by certain differential equations are characterized by (1) for appropriate choices of μ.

The study of generalized mean-values is not new, but previous authors have restricted themselves to measures of the form $\mu = \nu - \delta_0$, where ν is a positive measure of total mass 1 not concentrated on a hyperplane. In this case, the solutions to (1) are all real-analytic and satisfy a system of linear partial differential equations determined by ν. When all solutions of (1) are harmonic, the problem arises of characterizing the solution space of (1) explicitly. Problems of this sort were first studied by Walsh [37] and have been considered by numerous authors, including Beckenbach-Reade [4], [5], Brödel [9], Choquet-Deny [11], Flatto [15–18], Friedman-Littman [19], Garsia [20], and Garsia-Rodemich [21]. The emphasis of the present work is in quite a different direction. Because of this, and because the measure μ is not assumed to have a special form, the intersection of our results with earlier work on these lines is quite small. Our work also connects with some recent activity [1], [3], [14], [24], [33] in the area of functional equations; the overlap is minor and, at any rate, our approach is more systematic and our results considerably more general.

The paper is organized as follows. Section 1 contains a generalization of Pizzetti's formula [27], [28], interesting in its own right, which will be of use in the sequel. Section 2 deals with the problem of characterizing function classes by conditions like (1). We prove that, for a given μ, (1) is equivalent to an infinite collection of linear partial differential equations

$$Q_n(D)u = 0 \qquad n = 0, 1, 2, \cdots$$

where each Q_n is a homogeneous polynomial of degree n and solutions are understood in the weak (distributional) sense. If $P(\xi_1, \cdots, \xi_n)$ is a homogeneous polynomial, there exists a measure μ such that $u \in C(\mathscr{D})$ (or, more generally, $u \in L^1_{\mathrm{loc}}(\mathscr{D})$) is a weak solution to $P(D)u = 0$ if and only if (1) holds. The choices $P(\xi_1, \xi_2) = \xi_1^2 + \xi_2^2$ and $P(\xi_1, \xi_2) = \xi_1 + i\xi_2$ yield the results for harmonic and analytic functions mentioned above. Although the measure μ is by no means unique, it is easy to characterize the class of measures having the required property. In Section 3, we discuss our results in the context of the "two-circle" theorems of [40] and indicate how they extend to this situation. Related ideas are developed in Section 4, where we treat a question concerning the vanishing of certain Fourier coefficients; these results, which are related to those of the preceding section, were mentioned briefly in [40]. Section 5 concludes the paper with remarks on a natural modification of condition (1) and some indications for future research.

We employ the standard multi-index notation. Thus, $z = (z_1, \cdots, z_n)$, $\alpha = (\alpha_1, \cdots, \alpha_n)$, $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$, and so forth. The Fourier-Laplace transform of a distribution T of compact support is given by $\hat{T}(z) = \langle T, e^{-i(z\cdot\xi)}\rangle$. The symbolic vector $(-i\partial/\partial x_1, \cdots, -i\partial/\partial x_n)$ is denoted by D. With minor exceptions, our notation follows that of [22]. For the theory of linear partial differential equations with constant coefficients, see [22] and [34]. Treatments of the deeper properties of these equations, especially the exponential representation of solutions to systems, are found in [13] and [26]; a closely allied reference is [6]. The standard reference on distributions is [30]. We shall also require some basic facts from the algebra of polynomials; everything we need on this line may be found in [35].

Several of the results of this paper have been announced at various times in the Notices of the American Mathematical Society.

1.

Let μ be a finite complex Borel measure of compact support on $\mathbb{R}^n$ and set

$$F(z) = \int_{\mathbb{R}^n} e^{-i(z\cdot t)} d\mu(t),$$

the Fourier-Laplace transform of μ.

THEOREM 1. *Let $\mathscr{D}$ be a domain in $\mathbb{R}^n$ and u a real-analytic function defined on $\mathscr{D}$. Then for each $x \in \mathscr{D}$ we have*

$$\int u(x + rt)d\mu(t) = [F(-rD)u](x) \tag{2}$$

for all $r > 0$ for which the left-hand side exists and the right-hand side converges.

PROOF. The right-hand side is, of course, to be interpreted operationally. Since μ has compact support, the left-hand side exists for small r and is, moreover, obviously analytic in r. Since F is of exponential type and μ is real-analytic, the right-hand side also exists and is analytic for r sufficiently small. It is therefore enough to prove equality for small r. We have

$$\int e^{-i(z\cdot t)}d\mu(t) = \sum_{k=0}^{\infty} \frac{1}{k!} \int [-i(z\cdot t)]^k d\mu(t)$$
$$= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left\{ \int \sum_{|\alpha|=k} \binom{k}{\alpha} z^\alpha t^\alpha d\mu(t) \right\}$$
$$= \sum_{k=0}^{\infty} \frac{(-i)^k}{k!} \left\{ \sum_{|\alpha|=k} \binom{k}{\alpha} z^\alpha \int t^\alpha \, d\mu(t) \right\}$$
$$= \sum_{\alpha} \frac{1}{\alpha!}(-iz)^\alpha \left\{ \int t^\alpha d\mu(t) \right\}.$$

On the other hand, if $u(x+\xi) = \sum_\alpha a_\alpha(x)\xi^\alpha$,

$$\int u(x+rt)d\mu(t) = \int \left\{ \sum_\alpha a_\alpha(x) r^{|\alpha|} t^\alpha \right\} d\mu(t)$$
$$= \sum_\alpha a_\alpha(x) r^{|\alpha|} \left\{ \int t^\alpha d\mu(t) \right\}$$
$$= \sum_\alpha \frac{1}{\alpha!} (iD)^\alpha u(x) \cdot r^{|\alpha|} \left\{ \int t^\alpha d\mu(t) \right\}$$

since $a_\alpha(x) = (1/\alpha!)\,(\partial^{|\alpha|}/\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n})u(x)$.

Particular cases of this result are well known. Thus, when $n = 2$ the choice $d\mu = (1/2\pi)\,d\theta$ on the unit circle gives rise to

$$(3) \qquad \frac{1}{2\pi} \int_0^{2\pi} u(z + re^{i\theta})d\theta = [J_0(r\sqrt{-\Delta})u](z) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2} \left(\frac{r}{2}\right)^{2n} \Delta^n u(z),$$

the familiar formula of Pizzetti [27], [28]. (Compare the incorrect spellings of Courant-Hilbert [12, pp. 287, 812] and Nicolesco [25, pp. 7, 50].) Here, of course, the Fourier transform of μ is given by the Bessel function $J_0(\sqrt{z_1^2 + z_2^2})$. If we choose μ to be normalized Lebesgue area on the unit disc, the corresponding Fourier transform is $2J_1(\sqrt{z_1^2 + z_2^2})/\sqrt{z_1^2 + z_2^2}$ and we obtain

$$\frac{1}{\pi} \int_0^{2\pi} \int_0^1 u(z + r\rho e^{i\theta})\rho d\rho d\theta = \sum_{n=0}^{\infty} \frac{1}{n!\,(n+1)!} \left(\frac{r}{2}\right)^{2n} \Delta^n u(z).$$

By way of analogy, we shall call (2) the *generalized Pizzetti formula.*

The Pizzetti formula can be used to generate mean-value theorems for solutions to certain differential equations. Classical results in this direction include the identity

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta = \sum_{n=0}^{N} \frac{1}{(n!)^2}\left(\frac{r}{2}\right)^{2n} \Delta^n u(z), \tag{4}$$

valid for solutions of the polyharmonic equation $\Delta^{N+1}u = 0$, and Weber's relation [39]

$$\frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta = J_0(r\sqrt{c})u(z), \tag{5}$$

which holds for solutions of the Helmholtz equation $\Delta u + cu = 0$. These formulae admit considerable generalization. Indeed, each choice of measure $d\mu(\rho e^{i\theta}) = d\nu(\rho) \times d\theta/2\pi$ concentrated on the unit disc generates a formula

$$\int u(z+rt)d\mu(t) = \sum_{n=0}^{\infty} \frac{1}{(n!)^2}\left(\frac{r}{2}\right)^{2n} A_n\Delta^n u(z) \tag{6}$$

where $A_n = \int_0^1(\rho/2)^{2n}d\nu(\rho)$, and each of these, in turn, gives rise to analogues of (4) and (5). In dimensions higher than 2, similar formulae hold.

We should mention further that (6) in particular leads to an infinite number of mean-value theorems for polyharmonic functions. Indeed, let $(a_0, a_1, \cdots, a_N)$ be a solution of the system

$$\sum_{j=0}^{N} a_j r_j^{2k} = \delta_{0k} \qquad k = 0, 1, \cdots, N.$$

If we denote the point mass at $2r_j$ by δ_j, the product measure

$$\mu = \left(\sum_{j=0}^{N} a_j\delta_j(\rho)\right) \times d\theta/2\pi$$

satisfies

$$\int u(z+r\rho e^{i\theta})d\mu(\rho,\theta) = u(z)$$

for every function satisfying $\Delta^{N+1}u = 0$ and all sufficiently small r. Of course, similar formulae obtain with other choices of measures. For a fairly detailed study of related questions, see Poritsky [29].

A final application of Pizzetti's formula and its higher dimensional analogues concerns differential equations satisfied by certain mean-value operators. Thus, it is obvious from (3) that for real-analytic u

$$I(z,r) = \frac{1}{2\pi}\int_0^{2\pi} u(z+re^{i\theta})d\theta$$

satisfies the two-dimensional Euler-Poisson-Darboux equation

$$\frac{\partial^2 I}{\partial r^2} + \frac{1}{r}\ \frac{\partial I}{\partial r} = \Delta_z I,$$

where $\Delta_z = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Laplacian. Appropriate choices of μ in Theorem 1 lead to similar differential relationships. Once the differential equation is known, it can then be verified for (nonanalytic) functions with the appropriate degree of smoothness.

2.

Suppose now that u is real-analytic and satisfies (1). Then by the generalized Pizzetti formula

$$\sum_{\alpha} (-1)^{|\alpha|} r^{|\alpha|} A_\alpha D^\alpha u(x) = 0, \tag{7}$$

where $F(z) = \Sigma_\alpha A_\alpha z^\alpha$ is the Fourier-Laplace transform of μ. Equivalently,

$$Q_n(D)u = 0 \qquad n = 0,1,2,\cdots \tag{8}$$

where $Q_n(D) = \Sigma_{|\alpha|=n} A_\alpha D^\alpha$. It is thus clear that a necessary and sufficient condition for a real-analytic function u to satisfy (1) is that u be a solution to the system (8). The corresponding condition when no smoothness assumptions are made on u is most conveniently formulated in terms of weak solutions.

THEOREM 2. *The function $u \in C(\mathscr{D})$ satisfies* (1) *for each $x \in \mathscr{D}$ and all r such that $0 < r < \operatorname{dist}(x, \partial\mathscr{D})$ if and only if u is a weak solution in $\mathscr{D}$ of the system* (8).

PROOF. For convenience, set $u = 0$ off $\mathscr{D}$. If ϕ is a C^∞ function of compact support in $\mathscr{D}$, we have for each integer $N \geqq 0$

$$\int \phi(x - rt)d\mu(t) = \sum_{n=0}^{N} r^n Q_n(D)\phi(x) + o(r^N)$$

uniformly for $x \in \mathscr{D}$. Thus, if u satisfies (1), we have, for $0 < r < \operatorname{dist}(\operatorname{supp}\phi, \partial\mathscr{D})$,

$$\begin{aligned} 0 &= \int \left\{ \int u(x + rt)d\mu(t) \right\} \phi(x)dx \\ &= \int u(x) \left\{ \int \phi(x - rt)d\mu(t) \right\} dx \\ &= \int u(x) \left\{ \sum_{n=0}^{N} r^n Q_n(D)\phi(x) + o(r^N) \right\} dx \end{aligned}$$

for each $N \geqq 0$. Choosing N to be the smallest integer for which Q_N does not vanish identically, dividing by r^N, and making $r \to 0$, we obtain

$$\int u(x)Q_N(D)\phi(x)dx = 0. \tag{9}$$

It now follows by induction that (9) holds for $N = 0,1,2,\cdots$ so that u is a weak solution of (8).

The converse lies considerably deeper. According to the Hilbert Basis Theorem, the polynomial ideal generated by the $Q_n(\xi)$ $(n = 0,1,2,\cdots)$ is finitely generated. Thus the system (8) is equivalent to the *finite* system

$$Q_n(D)u = 0 \qquad n = 0,1,\cdots,N \tag{10}$$

for some integer N. Now fix $x \in \mathscr{D}$ and let $\Delta(x)$ be the largest ball about x lying in $\mathscr{D}$. It follows from a very deep and important theorem of Ehrenpreis and Palamodov, [13] and [26], on the exponential representation of solutions of systems of linear partial differential equations with constant coefficients that each solution of (10) may be approximated uniformly on compact subsets of $\Delta(x)$ by *real-analytic* solutions of (10). From this, it is clear that (1) must hold for all $r < \text{dist}\,(x, \partial\mathscr{D})$.

In certain situations, the system (8) is completely redundant and may be replaced by a single equation of the form $P(D)u = 0$, where P is a homogeneous polynomial. This is the case, for instance, when $\mu = \Omega - \delta_0$ and gives rise to the fact that the mean-value property is a necessary as well as a sufficient condition for harmonicity. The general situation is described in the following result.

THEOREM 3. *The system* (8) *is equivalent to*

$$P(D)u = 0 \tag{11}$$

if and only if the (*homogeneous*) *polynomial P divides each of the polynomials Q_n and, for some n, $P = cQ_n$, c a constant.*

Proof. Half of the theorem is trivial. Suppose then that the system (8) and the single equation (11) have the same solutions. Fix n such that $Q_n \not\equiv 0$ and write $P = P_1\tilde{P}$, $Q_n = \tilde{Q}_n\tilde{P}$, where $\tilde{P}$ is the greatest common factor of P and Q_n (unique up to a multiplicative constant). For each $z \in C^n$ the equation

$$\tilde{P}(D)v = \exp\left[i(z\cdot\xi)\right]$$

may be solved on $\mathbb{R}^n$ [22, p. 82]. Hence, if $P_1(z) = 0$, we have

$$\begin{aligned} P(D)v = P_1(D)\tilde{P}(D)v &= P_1(D)\{\exp\left[i(z\cdot\xi)\right]\} \\ &= P_1(z)\exp\left[i(z\cdot\xi)\right] = 0, \end{aligned}$$

so that $Q_n(D)v = \tilde{Q}_n(z) \exp[i(z \cdot \xi)] = 0$. Thus, $\tilde{Q}_n$ vanishes whenever P_1 does, so that (Nullstellensatz!) P_1 divides a power of $\tilde{Q}_n$. Since P_1 and $\tilde{Q}_n$ are relatively prime, P_1 is constant and so P divides Q_n: $Q_n = \tilde{Q}_n P$. If the degree of $\tilde{Q}_n$ is greater than zero for all n, any solution of $P(D)u = 1$ satisfies (8) but not (11). Hence $Q_n = cP$ for some n, and it is clear that this is the smallest index for which Q_n does not vanish identically.

Now let $P(\xi)$ be a fixed homogeneous polynomial. Combining Theorem 3 with (2) and (7), we see that (1) is equivalent to $P(D)u = 0$ if and only if $\hat{\mu}(z) = P(z)h(z)$, where h is entire and $h(0) \neq 0$. By the Paley-Wiener Theorem [22, p. 21], $h(z) = \hat{T}(z)$ for some distribution T of compact support and $\mu = P(D)T$. For a concrete realization, let χ be any sufficiently smooth function supported in the unit ball having nonvanishing integral, and set $d\mu = P(D)\chi(t)dt$. Summarizing, we have

THEOREM 4. *For each homogeneous polynomial P there exists a finite measure μ of compact support such that for $\mathscr{D} \subset \mathbb{R}^n$, $u \in C(\mathscr{D})$ is a weak solution of $P(D)u = 0$ if and only if*

$$\int u(x + rt)d\mu(t) = 0$$

for all $x \in \mathscr{D}$ and $0 < r < \operatorname{dist}(x, \partial\mathscr{D})$. In fact, any measure of the form $P(D)T$, where T is a distribution of compact support such that $\hat{T}(0) \neq 0$, *has the required property.*

Of course, not every such measure need arise from a smooth function; indeed formulae involving singular measures are often of particular interest. Although the most familiar examples of such formulae occur in the theory of analytic and harmonic functions, the phenomenon is by no means restricted to such special classes. Thus, the weak solutions to

$$\frac{\partial^n u}{\partial z^n} \pm \frac{\partial^n u}{\partial \bar{z}^n} = 0 \tag{12}$$

are characterized by the condition

$$\int_0^{2\pi} u(x + r\cos\theta,\ y + r\sin\theta) \begin{Bmatrix} \cos\ n\theta \\ \sin\ n\theta \end{Bmatrix} d\theta = 0.$$

When $n = 2$, (12) becomes d'Alembert's equation $u_{xx} - u_{yy} = 0$; the corresponding mean-value condition was observed by Shapiro [31] in a rather different context

3.

In [40], we showed how the theory of mean-periodic functions of one variable could be combined with distributional Fourier transform techniques to characterize functions obeying certain integral conditions on two circles (of fixed radius) about each point in the complex plane as solutions of equations of the form $\partial^{n+m}f/\partial z^n\partial\bar{z}^m = 0$. It is easy to cast Theorem 4 into a similar mold. Let μ be a finite measure of compact support and let $S = S(\mu)$ be an at most countable set of real numbers. We say that μ determines the differential equation $P(D)u = 0$ if the condition

$$\int u(x + rt)d\mu(t) = 0 \tag{13}$$

for almost all $x \in \mathbb{R}^n$ and $r = r_1, r_2$ is equivalent to $P(D)u = 0$ (weakly) whenever $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $r_1/r_2 \notin S$.

THEOREM 5. *For each homogeneous polynomial $P(\xi_1,\cdots,\xi_n)$ there exists a compact measure μ, absolutely continuous with respect to volume, which determines the differential equation $P(D)u = 0$.*

PROOF. Let P be given. Let χ be a smooth radial function of compact support which has nonzero integral and set $d\mu = P(D)\chi(t)dt$. Then $\hat{\mu}(z) = P(z)F(z)$, where F is the Fourier transform of χ. Now F is an even entire function of the complex variable $\zeta = \sqrt{z_1^2 + \cdots + z_n^2}$; thus, writing $F(z) = \tilde{F}(\zeta)$, we have $\tilde{F}(0) = F(0) \neq 0$. We may rewrite (13) as the convolution pair

$$\mu_1 * u = 0 \qquad \mu_2 * u = 0$$

where $\hat{\mu}_j(z) = \hat{\mu}(-r_jz)$. Let $S(\mu) = \{\zeta_1/\zeta_2 \colon \tilde{F}(\zeta_1) = \tilde{F}(\zeta_2) = 0\}$. Then if $r_1/r_2 \notin S$, $F(r_1z)$ and $F(r_2z)$ have no common zeroes. Proceeding as in [40], we see that if $u \in L^1_{\text{loc}}(\mathbb{R}^n)$ satisfies (13) it must be a distributional solution to $P(D)u = 0$. Conversely, if $P(D)u = 0$, then for each measure ν of the form $P(D)T$ we have $\nu * u = P(D)T * u = T * P(D)u = 0$. Since P is homogeneous, the measures μ_j have the required form, so that (13) holds for all r.

It is easy to see that the preceding result is limited to homogeneous polynomials. For suppose the differential equation $P(D)u = 0$ is determined by μ and let u be a continuous solution of $P(D)u = 0$. It then follows that (13) must hold for all x and r, and the discussion of the previous section shows that P must actually be a homogeneous polynomial. On the other hand, there are nonhomogeneous differential polynomials whose solutions satisfy conditions like (13) for certain thin sets of r. An example is already provided in $\mathbb{R}^2$ by the equation $\Delta u + cu = 0$ $(c > 0)$, $d\mu(e^{i\theta}) = d\theta$. Comparing (13) with (5), we see that (13) is satisfied for $r = x_n / \sqrt{c}$, where $\{x_n\}$ is the collection of positive zeroes of the Bessel function $J_0(z)$.

4.

Our concluding result, which arose in response to an inquiry of Jean-Pierre Rosay, introduces a new variation on the theme of [40].

THEOREM 6. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^2)$ and let $r > 0$ be fixed. Suppose there exist integers n, m such that for almost all $z \in \mathbb{C}$*

$$\tag{14} \begin{aligned} &\int f(z + re^{i\theta})e^{in\theta}\, d\theta = 0 \\ &\int f(z + re^{i\theta})e^{im\theta}\, d\theta = 0. \end{aligned}$$

Then

(a) *if $0 \leqq n < m$, f agrees almost everywhere with a solution of $\partial^n f / \partial \bar{z}^n = 0$;*

(b) *if $0 \geqq n > m$, f agrees almost everywhere with a solution of $\partial^{|n|} f / \partial z^{|n|} = 0$;*

(c) *if $n > 0 > m$, $m \neq -n$, f agrees almost everywhere with a solution of the pair of equations $\partial^n f / \partial \bar{z}^n = 0$, $\partial^{|m|} f / \partial z^{|m|} = 0$. Thus, in this case f is (essentially) a polynomial.*

Here solutions are understood in the strong sense.

For the proof we need the following

LEMMA. *Let n and m be distinct integers, $m \neq -n$. The Bessel functions $J_n(z)$ and $J_m(z)$ have no common zeroes other than $z = 0$.*

Of course, when n and m differ by one, this result is classical. The assertion of the lemma (known as Bourget's hypothesis, after J. Bourget, who conjectured it in [7]) is considerably more difficult and lies much deeper. It follows from the result of Carl Ludwig Siegel [32] that the zeroes of $J_\alpha(z)$ are transcendental whenever α is algebraic; cf. [38, pp. 484–485].

PROOF. We prove (a) and (c); (b) follows from (a) by complex conjugation. In case $0 \leqq n < m$, the Fourier transforms of the measures $\mu_k = e^{ik\theta} d\theta$ restricted to the circle $|z| = r$ are given by

$$\hat{\mu}_k(z_1, z_2) = 2\pi(z_2 - iz_1)^k J_k(r\sqrt{z_1^2 + z_2^2}) / (\sqrt{z_1^2 + z_2^2})^k \qquad k = n, m.$$

From (14), which may be written as

$$\mu_n * f = 0, \qquad \mu_m * f = 0,$$

we obtain

$$\frac{\partial^n T_n}{\partial \bar{z}^n} * f = 0, \ \frac{\partial^n T_m}{\partial \bar{z}^n} * f = 0$$

or, equivalently,

$$T_n * \frac{\partial^n f}{\partial \bar{z}^n} = 0, \ T_m * \frac{\partial^n f}{\partial \bar{z}^n} = 0. \tag{15}$$

Here derivatives are taken in the distributional sense and the distributions T_n, T_m of compact support are determined by

$$\hat{T}_n(z_1, z_2) = 2\pi J_n(r\sqrt{z_1^2 + z_2^2})/(\sqrt{z_1^2 + z_2^2})^n$$

$$\hat{T}_m(z_1, z_2) = 2\pi(z_2 - iz_1)^{m-n} J_m(r\sqrt{z_1^2 + z_2^2})/\left(\sqrt{z_1^2 + z_2^2}\right)^n.$$

Since $\hat{T}_n$ and $\hat{T}_m$ have no common zeroes, it follows as in [40] that (15) implies $\partial^n f / \partial \bar{z}^n = 0$ as a distribution. Weyl's lemma then yields the assertion in (a).

The proof of (c) is only slightly more complicated. The Fourier transforms of the measures $\mu_n = e^{in\theta} d\theta$, $\mu_m = e^{im\theta} d\theta$ on $|z| = r$ are now given by

$$\hat{\mu}_n(z_1, z_2) = 2\pi(z_2 - iz_1)^n J_n(r\sqrt{z_1^2 + z_2^2})/(\sqrt{z_1^2 + z_2^2})^n$$

$$\hat{\mu}_m(z_1, z_2) = 2\pi(z_2 + iz_1)^{|m|} J_m(r\sqrt{z_1^2 + z_2^2})/(\sqrt{z_1^2 + z_2^2})^{|m|}.$$

Since $\mu_n * f = 0$, $\mu_m * f = 0$, we have

$$\frac{\partial^n T_n}{\partial \bar{z}^n} * f = 0, \ \frac{\partial^n \mu_m}{\partial \bar{z}^n} * f = 0$$

or

$$T_n * \frac{\partial^n f}{\partial \bar{z}^n} = 0, \ \mu_m * \frac{\partial^n f}{\partial \bar{z}^n} = 0$$

where T_n is as before. Since $\hat{T}_n$ and $\hat{\mu}_m$ have no common zeroes, $\partial^n f / \partial \bar{z}^n = 0$. A similar argument shows $\partial^{|m|} f / \partial z^{|m|} = 0$.

Theorem 5 has an attractive (and obvious) interpretation in terms of the Fourier coefficients of the restriction of f to circles of radius r. It is sharp in the sense that the result fails in case $n = -m$ or only one Fourier coefficient is specified; examples are easily constructed along the lines indicated in [40]. Of particular note here is the fact that, unlike previous results, no exceptional set extrudes itself into the present theorem.

5.

In closing, we wish to remark briefly on the condition

$$\lim_{r\to 0} \frac{1}{r^k} \int u(x+rt)d\mu(t) = 0, \tag{16}$$

where $u \in C(\mathscr{D})$ and (16) is to be satisfied pointwise in $\mathscr{D}$. Suppose μ is orthogonal to all polynomials of total degree less than k and that $u \in C^k(\mathscr{D})$. It is then easy to see that (16) is equivalent to the single equation $P(D)u = 0$, where P is the homogeneous polynomial of degree k which is the first nonvanishing term in the Taylor expansion for $\hat{\mu}(z)$.

For functions having less smoothness, the situation is rather different and seems to deserve fuller investigation. For instance, let $d\mu(t) = g(\rho)d\rho\sin\theta d\theta$, where $t = \rho e^{i\theta}$ and

$$\int_0^1 g(\rho)\rho d\rho = 0, \int_0^1 g(\rho)\rho^3 d\rho \neq 0.$$

Equation (16) then characterizes the solutions of

$$\frac{\partial}{\partial y}\Delta u = 0. \tag{17}$$

Moreover, the function $u(x,x) = |y|$ satisfies

$$\lim_{r\to 0} \frac{1}{r^3} \int_0^1 \int_0^{2\pi} u(x + r\rho\cos\theta,\ y + r\rho\sin\theta)\sin\theta\ d\theta\ g(\rho)d\rho = 0$$

for all (x,y), but is not distributional solution of (17); see [10] for a related example. More general examples are easily constructed along the above lines; a C^{k-3} function satisfying (16) may fail to satisfy the associated differential equation. In a related connection, Arsove [2] has shown, using a result of Kurt Meier, that for $\mathscr{D} \subset \mathbb{C}$

$$\lim_{r\to 0} \frac{1}{r} \int_{|\zeta|=1} u(z+r\zeta)d\zeta = 0 \qquad z \in \mathscr{D}$$

if and only if u is holomorphic in $\mathscr{D}$.

Finally, we take this opportunity to correct an attribution in [40]. There, in

Section 9, we followed Littlewood in crediting the one-circle converse to the mean value theorem for harmonic functions to Kellogg. Not surprisingly, the result is much older; it first appears, to the best of my knowledge, in a paper of Volterra [36].

References

1. J. Aczél, H. Haruki, M. A. McKiernan, and G. N. Sakovič, *General and regular solutions of functional equations characterizing harmonic polynomials*, Aequationes Math. **1** (1968), 37–53.

2. M. G. Arsove, *The Looman-Menchoff theorem and some subharmonic function analogues*, Proc. Amer. Math. Soc. **6** (1955), 94–105.

3. J. A. Baker, *An analogue of the wave equation and certain related equations*, Canad. Math. Bull. **12** (1969), 837–846.

4. E. F. Beckenbach and M. Reade, *Mean values and harmonic polynomials*, Trans. Amer. Math. Soc. **53** (1943), 230–238.

5. E. F. Beckenbach and M. Reade, *Regular solids and harmonic polynomials*, Duke Math. J. **12** (1945), 629–644.

6. C. A. Berenstein and M. A. Dostal, *Analytically Uniform Spaces and their Applications to Convolution Equations*, Springer-Verlag, 1972.

7. J. Bourget, *Mémoire sur le mouvement vibratoire des membranes circulaires*, Ann. Sci. Ecole Norm. Sup. **3** (1866), 55–95.

8. J. H. Bramble and L. E. Payne, *Mean value theorems for polyharmonic functions*, Amer. Math. Monthly **73** (1966), Part II, 124–127.

9. W. Brödel, *Funktionen mit Gaussischer Mittelwerteigenschaften für konvexe Kurven und Bereiche*, Deutsche Mathematik **4** (1939), 3–15.

10. Min-Teh Cheng, *On a theorem of Nicolesco and generalized Laplace operators*, Proc. Amer. Math. Soc. **2** (1951), 77–86.

11. G. Choquet and J. Deny, *Sur quelques propriétés de moyenne charactéristiques des fonctions harmoniques et polyharmoniques*, Bull. Soc. Math. France **72** (1944), 118–140.

12. R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Vol. 2, Interscience, 1962.

13. L. Ehrenpreis, *Fourier Analysis in Several Complex Variables*, Interscience, 1970.

14. I. Fenyö, *Remark on a paper of J. A. Baker*, Aequationes Math. **8** (1972), 103–108.

15. L. Flatto, *Functions with a mean value property*, J. Math. Mech. **10** (1961), 11–18.

16. L. Flatto, *Functions with a mean value property II*, Amer. J. Math. **85** (1963), 248–270.

17. L. Flatto, *Partial differential equations and difference equations*, Proc. Amer. Math. Soc. **16** (1965). 858–863.

18. L. Flatto, *On polynomials characterized by a certain mean value property*, Proc. Amer. Math. Soc. **17** (1966), 598–601.

19. A. Friedman and W. Littman, *Functions satisfying the mean value property*, Trans. Amer. Math. Soc. **102** (1962), 167–180.

20. A. Garsia, *A note on the mean value property*, Trans. Amer. Math. Soc. **102** (1962), 181–186.

21. A. M. Garsia and E. Rodemich, *On functions satisfying the mean value property with respect to a product measure*, Proc. Amer. Math. Soc. **17** (1966), 592–594.

22. L. Hörmander, *Linear Partial Differential Operators*, Springer-Verlag, 1964.

23. J. E. Littlewood, *A Mathematician's Miscellany*, Methuen, 1953.

24. M. A. McKiernan, *Boundedness on a set of positive measure and the mean value property characterizes polynomials on a space V^n*, Aequations Math. **4** (1970), 31–36.

25. M. Nicolesco, *Les Fonctions Polyharmoniques*, Hermann, 1936.

26. V. P. Palamodov, *Linear Differential Operators with Constant Coefficients*, Springer-Verlag, 1970.

27. P. Pizzetti, *Sulla media dei valori che una funzione dei punti dello spazio assume alla superficie di una sfera*, Rend. Lincei, ser. 5, 18 (1909), 182–185.

28. P. Pizzetti, *Sul significato geometrico del secundo parametro differenziale di una funzione sopra una superficie qualunque*, Rend. Lincei, ser. 5, **18** (1909), 309–316.

29. H. Poritsky, *Generalizations of the Gauss law of the spherical mean*, Trans. Amer. Math. Soc. **43** (1938), 199–225.

30. L. Schwartz, *Théorie des Distributions*, Vols. 1 and 2, Hermann, 1957.

31. V. L. Shapiro, *Sets of uniqueness for the vibrating string problem*, Trans. Amer. Math. Soc. **141** (1969), 127–146.

32. C. L. Siegel, *Über einige Anwendungen diophantischer Approximationen*, Abh. Preuss. Akad. Wiss. Phys-math. Kl. 1929, no. 1.

33. H. Światak, *Criteria for the regularity of continuous and locally integrable solutions of a class of linear functional equations*, Aequations Math. **6** (1971), 170–187.

34. F. Trèves, *Linear Partial Differential Equations with Constant Coefficents*, Gordon and Breach, 1966.

35. B. L. van der Waerden, *Modern Algebra*, Vols. 1 and 2, Frederich Ungar, 1953, and 1950.

36. V. Volterra, *Alcune osservazioni sopra proprietà atte ad individuare una funzione*, Rend. Lincei, ser. 5, **18** (1909), 263–266.

37. J. L. Walsh, *A mean value theorem for polynomials and harmonic polynomials*, Bull. Amer. Math. Soc. **42** (1936), 923–930.

38. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1962.

39. H. Weber, *Ueber die Integration der partiellen Differentialgleichung:* $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + k^2 u = 0$, Math. Ann. **1** (1869), 1–36.

40. L. Zalcman, *Analyticity and the Pompeiu problem*, Arch. Rat. Mech. Anal. **47** (1972), 237–254.

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